SHORT COMMUNICATIONS

Moments of the folded logistic distribution

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Abstract The recent paper by Cooray et al. introduced the folded logistic distribution. The moments properties given in the paper appear too complicated. In this note, a simple formula is derived in terms of the well known Lerch function.

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The recent paper by Cooray et al.[1] introduced a novel distribution referred to as the folded logistic distribution. The probability density function (PDF) and the cumulative distribution function (CDF) of this distribution are given as

$$f(x) = \frac{1}{\sigma}\exp\left(-\frac{x-\mu}{\sigma}\right)\left\{1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}^{-2} + \frac{1}{\sigma}\exp\left(-\frac{x+\mu}{\sigma}\right)\left\{1+\exp\left(-\frac{x+\mu}{\sigma}\right)\right\}^{-2} \quad (1)$$

and

$$F(x) = \left\{1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}^{-1} + \left\{1+\exp\left(-\frac{x+\mu}{\sigma}\right)\right\}^{-1} - 1 \quad (2)$$

respectively, for $x>0$, $-\infty<\mu<\infty$ and $\sigma>0$. The moments $E(X^r)$ (where X is a random variable with the PDF and the CDF specified by (1) and (2), respectively) are expressed as double infinite sums. In addition, separate expressions are given for r even and r odd. We feel that these expressions are unnecessarily complicated. In this paper, we show that one can derive a simple expression for $E(X^r)$—applicable for any r—in terms of the well known Lerch function defined by

$$\Phi(z, s, \nu) = \sum_{n=0}^{\infty} \frac{z^n}{(\nu+n)^s} \quad (3)$$

or equivalently

$$\Phi(z, s, \nu) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}\exp(-\nu t)}{1-z\exp(-t)}\mathrm{d}t \quad (4)$$

for $|z|<1$ and $\nu\neq 0, -1, -2, \cdots$. We refer the readers to Chapter 1 of Erdelyi et al.[2] and Section 9.55 of Gradshteyn and Ryzhik[3] for detailed properties of the Lerch function. Numerical routines for the computation of (3) and (4) are widely available, e.g. LerchPhi in Maple.

Theorem 1 gives the expression for $E(X^r)$ for any real $r>0$. Theorem 2 provides an equivalent expression for $E(X^r)$. Corollaries 1 and 2 consider some special cases.

Theorem 1. If X is a random variable with the pdf and the cdf specified by (1) and (2), respectively, then

$$E(X^r) = \sigma^r\Gamma(r+1)\left\{\exp\left(\frac{\mu}{\sigma}\right)\Phi\left(-\exp\left(\frac{\mu}{\sigma}\right), r, 1\right) + \exp\left(-\frac{\mu}{\sigma}\right)\Phi\left(-\exp\left(-\frac{\mu}{\sigma}\right), r, 1\right)\right\} \quad (5)$$

for any real $r>0$.

Proof. Using (2), one can write $E(X^r)$ as

$$\begin{aligned} E(X^r) &= r\int_0^\infty x^{r-1}\left[2-\left\{1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}^{-1} - \left\{1+\exp\left(-\frac{x+\mu}{\sigma}\right)\right\}^{-1}\right]\mathrm{d}x \\ &= r\int_0^\infty x^{r-1}\exp\left(-\frac{x-\mu}{\sigma}\right) \cdot \left\{1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}^{-1}\mathrm{d}x \\ &\quad + r\int_0^\infty x^{r-1}\exp\left(-\frac{x+\mu}{\sigma}\right) \end{aligned}$$

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$$\cdot\left\{1+\exp\left(-\frac{x+\mu}{\sigma}\right)\right\}^{-1}\mathrm{d}x \tag{6}$$

Setting $y=x/\sigma$, one can reexpress (6) as

$$E(X^r)=r\sigma^r\exp\left(\frac{\mu}{\sigma}\right)\int_0^\infty y^{r-1}\exp(-y)$$
$$\cdot\left\{1+\exp\left(\frac{\mu}{\sigma}\right)\exp(-y)\right\}^{-1}\mathrm{d}y$$
$$+r\sigma^r\exp\left(-\frac{\mu}{\sigma}\right)\int_0^\infty y^{r-1}\exp(-y)$$
$$\cdot\left\{1+\exp\left(-\frac{\mu}{\sigma}\right)\exp(-y)\right\}^{-1}\mathrm{d}y \tag{7}$$

The result in (5) follows by using the definition of the Lerch function given by (4). QED

Theorem 2. Under the conditions of Theorem 1, an equivalent expression for $E(X^r)$ is

$$E(X^r)=-\sigma^r\,\Gamma(r+1)\left\{\mathrm{Li}_r\left(-\exp\left(\frac{\mu}{\sigma}\right)\right)+\mathrm{Li}_r\left(-\exp\left(-\frac{\mu}{\sigma}\right)\right)\right\} \tag{8}$$

for $r>0$, where $\mathrm{Li}_r(\cdot)$ denotes the polylogarithm of order r defined by

$$\mathrm{Li}_r(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^r}$$

Proof. Follows from (5) by using the fact that $\mathrm{Li}_r(z)=z\Phi(z,r,1)$. QED

Corollary 1. Under the conditions of Theorem 1, if r is an even integer then

$$E(X^r)=\sigma^r(2\mathrm{i}\pi)^r B_r\left(\frac{\ln w}{2\mathrm{i}\pi}\right)$$

where $\mathrm{i}=\sqrt{-1}$, $w=-\exp(\mu/\sigma)$ and $B_m(\cdot)$ denotes the Bernoulli polynomial of order m defined by

$$B_m(x)=\sum_{n=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^m$$

Proof. follows from (8) by using the fact that $\mathrm{Li}_r(z)+\mathrm{Li}_r(1/z)=-(1/r!)(2\mathrm{i}\pi)^r B_r(\ln z/(2\mathrm{i}\pi))$. QED

Corollary 2. Under the conditions of Theorem 1, the expected value of X is given by

$$E(X)=\mu+2\sigma\ln\left\{1+\exp\left(-\frac{\mu}{\sigma}\right)\right\}$$

Proof. Follows from (8) by using the fact that $\mathrm{Li}_1(z)=-\ln(1-z)$. QED

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